$$\frac{1}{n}\ln(1+\frac{k}{n}) \le \int_{1+\frac{k}{n}}^{1+\frac{k+1}{n}} \ln x \quad dx \le \frac{1}{n}\ln(1+\frac{k+1}{n})$$

$$\Rightarrow \frac{1}{n}\sum_{k=0}^{n-1}\ln(1+\frac{k}{n}) \le \sum_{k=0}^{n-1}\int_{1+\frac{k}{n}}^{1+\frac{k+1}{n}} \ln x \quad dx \le \frac{1}{n}\sum_{k=0}^{n-1}\ln(1+\frac{k+1}{n})$$

$$\text{or } \frac{1}{n}\sum_{k=0}^{n-1}\ln(1+\frac{k}{n}) = \frac{1}{n}(\ln 1 + \sum_{k=1}^{n}\ln(1+\frac{k}{n}) - \ln(1+\frac{n}{n}) = U_n - \frac{1}{n}\ln 2$$

• Montrons que $\frac{1}{n}\sum_{k=0}^{n-1}\ln(1+\frac{k+1}{n}) = U_n$.

$$\frac{1}{n}\sum_{k=0}^{n-1}\ln(1+\frac{k+1}{n}) = \frac{1}{n}(\ln(1+\frac{1}{n}) + \ln(1+\frac{2}{n}) + \cdots + \ln(1+\frac{n}{n})) = U_n$$

• $$\sum_{k=0}^{n-1}\int_{1+\frac{k}{n}}^{1+\frac{k+1}{n}} \ln x \quad dx = \int_{1}^{1+\frac{1}{n}} \ln x \quad dx + \int_{1+\frac{1}{n}}^{1+\frac{2}{n}} \ln x \; dx + \cdots$$

$$\cdots + \int_{1+\frac{n-2}{n}}^{1+\frac{n-1}{n}} \ln x \quad dx + \int_{1+\frac{n-1}{n}}^{1+\frac{n}{n}} \ln x \quad dx = \int_{1}^{2} \ln x \quad dx$$

Conclusion : $\forall n \in \mathbf{N}^*$, $U_n - \frac{1}{n}\ln 2 \le \int_1^2 \ln x \; dx \le U_n$.

b) On a : $U_n - \frac{1}{n}\ln 2 \le \int_1^2 \ln x \quad dx$

donc $U_n \le \frac{1}{n}\ln 2 + \int_1^2 \ln x \quad dx$

d'où $\forall n \in \mathbf{N}^*$, $\int_1^2 \ln x \quad dx \le U_n \le \frac{1}{n}\ln 2 + \int_1^2 \ln x \quad dx$

$$\lim_{n\to+\infty} \frac{1}{n}\ln 2 + \int_1^2 \ln x \; dx = 0 + \int_1^2 \ln x \quad dx = \int_1^2 \ln x \quad dx$$

d'où $\lim_{n\to+\infty} U_n = \int_1^2 \ln x \quad dx = \left[x \quad \ln x - x\right]_1^2 = 2 \quad \ln 2 - 1 = l$.

Conclusion : (U_n) est une suite convergente vers $l = 2\ln 2 - l$.